If you eat a meal consisting of 9 ounces of meat, 20 ounces of potatoes, and 5 ounces of cabbage, how many grams of each nutrient do you get? Use matrix multiplication to obtain your answer.

10. ⟨1⟩ If A is a column vector of n elements and B is a row vector of n elements, describe the form of AB.

11. ⟨1⟩ Consider the four matrices

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad D = \begin{bmatrix} \sin\theta & \cos\theta \\ \cos\theta & -\sin\theta \end{bmatrix}.$$

Show that the square of each one of them is I. This shows that the identity matrix, unlike the identity number, can have many square roots.

12. ⟨2⟩ Let

$$M = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Compute M^2 and M^3. What do you conjecture about the form of M^n?

13. ⟨1⟩ Express the following system of equations as one matrix equation.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ 2x_1 - 3x_2 + 4x_3 &= 5. \end{aligned}$$

14. ⟨2⟩ Let

$$\begin{aligned} c &= b_1 + 2b_2 + 3b_3; \\ b_1 &= a_1 + 2a_2; \\ b_2 &= 3a_1 + 4a_2; \\ b_3 &= 5a_1 + 6a_2. \end{aligned}$$

Express c in terms of a_1 and a_2 two ways:

a) By direct algebraic substitution.

b) By matrix multiplication.

Show how various calculations in the two methods correspond (say, by drawing arrows between corresponding parts).

15. ⟨2⟩ What sizes must A, B, and C be for ABC to exist? Assuming it exists, write a formula using summation notation for the typical entry of ABC. Start by letting $A = [a_{ij}]$.

0.6 The Language and Methods of Reasoning

Our purposes in this section are to

1. Make sure you understand the key words used in mathematical reasoning, like "necessary", "hypothesis", and "any";
2. Introduce some logic notation, like $\Longrightarrow$;
3. Discuss some basic methods of proof.

Our approach is informal. In Chapter 7 we'll introduce a formal algebra of logic and use it to (among other things) review and further clarify all the points about logic and proof we will have made.

Implication

The most common type of sentence in a mathematical argument is an **implication**. This is anything of the form A implies B, that is, *if* A *then* B. For instance,

$$\text{if } x = 2, \text{ then } x^2 = 4.$$

The if-part of the sentence, $x = 2$, is called the **hypothesis**. Synonyms are **assumption** and **premise**. The then-part, $x^2 = 4$, is called the **conclusion**.

Sometimes an implication is broken into two or more sentences. Indeed, this is quite common when the implication is asserted as a theorem. For instance, you might find

Theorem. Let n be an odd integer. Then n^2 is also odd.

This theorem is really the assertion of the implication

If n is an odd integer, then n^2 is odd.

Also, neither the hypothesis nor the conclusion of an implication needs to be a simple statement. For instance, consider the theorem

Let ABC be a triangle. Suppose angle A = angle B. Suppose further that angle $C = 60°$. Then ABC is equilateral. Furthermore, all three angles are $60°$.

In this theorem the first three sentences together form the hypothesis and the last two are the conclusion.

Sometimes an implication, rather than expanding from two clauses (if and then) to several sentences, instead shrinks down to a single clause. Consider the statement

All squares are rectangles.

This is really an implication:

If a quadrilateral is a square, then it is a rectangle.

Instead of "quadrilateral", we could have said "figure" or "something". It turns out it doesn't make any difference.

There is a nice notation for A implies B:

$$A \Longrightarrow B.$$

Some authors write $A \rightarrow B$ instead, but single-shaft arrows are also used for limits of sequences and functions, e.g., $x_n \rightarrow 0$ as $n \rightarrow \infty$ (see our use of this notation in Sections 0.2 and 0.3). We prefer to keep the different uses distinct. In this book, $\Longrightarrow$ is a **logical operator** making a single statement from two separate statements, whereas $\rightarrow$ is an abbreviation for the verb "approaches".

(In fact, one can distinguish between several strengths of implication, and in some books $\Longrightarrow$ is only used for one of them. For instance, $(x = 2) \Longrightarrow (x^2 = 4)$ is a stronger implication than "if this is the Sears tower, then this is the tallest building in the world." Although the former claim is always true, the latter switched from true to false between the first and second editions of this book. At any rate, we will not distinguish between levels of implication and we use $\Longrightarrow$ for all of them.)

One reason implications are so useful in mathematics is because they chain together so nicely. That is, if $A \Longrightarrow B$ and $B \Longrightarrow C$ are both true, then clearly so is $A \Longrightarrow C$, and so on with longer chains. This transitivity property (see Relations in Section 0.1) means that, using implications, a mathematician can try to show that A implies Z by reducing it to many smaller steps.

Many different English phrases boil down to $\Longrightarrow$. We have already mentioned "implies" and "if–then". Here are some others. Each sentence below means $A \Longrightarrow B$.

Whenever A is true, so is B.

B follows from A.

A is sufficient for B.

B is necessary for A.

A only if B.

It's not supposed to be obvious that all of these mean $A \Longrightarrow B$, especially when stated so abstractly. So, first, let's restate them with A = "It is raining" and B = "It is cloudy".

Whenever it's raining, it's cloudy.

It follows that it's cloudy, if it's raining.

It's sufficient that it rain in order that it be cloudy.

It is necessary that it be cloudy for it to rain.

It's raining only if it's cloudy.

We hope you see that they mean essentially the same thing as "if it's raining, then it's cloudy". There is a danger in using ordinary English sentences as examples. First, some of these constructions make awkward English. Second, different constructions in English (or any natural language) have different nuances, so you may not feel that these sentences mean exactly the same thing. But we hope you agree that they have the same "logical" content.

Did it surprise you that, in some of these equivalent statements, the order of appearance of A and B in the sentence changed? It may seem particularly odd that "A only if B" should mean the same as "if A, then B" – the "if" has moved from A to B! But that's the way it is – natural languages are full of oddities.

The use of "necessary" and "sufficient" is particularly common in mathematics. We'll return to these words in a moment.

Closely related to the implication $A \Longrightarrow B$ is its **converse**, $B \Longrightarrow A$. (This is sometimes written $A \Longleftarrow B$.) The converse has a separate name because it does *not* mean the same thing as the original. For instance,

If it's raining, then it's cloudy

is true, but

If it's cloudy, then it's raining

is false because it can be cloudy without raining.

We hope you are already familiar with the difference between a statement and its converse. If you are told to assume A and prove B, you must demonstrate $A \Longrightarrow B$; if you demonstrate $B \Longrightarrow A$ instead, you have wasted your time, because $A \Longrightarrow B$ could still be false. If you start with A and end up with B, you have done the job. If instead you start with B and end up with A, you have shown $B \Longrightarrow A$ and goofed. This is called *assuming what you are supposed to show.*

There is another implication closely related to $A \Longrightarrow B$ which *does* mean the same thing: its **contrapositive**, i.e.,

$$(\textbf{not } B) \implies (\textbf{not } A),$$

or using the notation for "not" that we'll use in Chapter 7,

$$\neg B \implies \neg A.$$

In the contrapositive, the order of statements is reversed and they are both negated. Thus the contrapositive of

If it's raining, then it's cloudy

is

If it's not cloudy, then it's not raining.

When we say an implication and its contrapositive mean the same thing, we mean that either they are both true or they are both false.

Let's return to the words "necessary" and "sufficient". Remember,

$A \Longrightarrow B$ means the same as A is *sufficient* for B and also the same as B is *necessary* for A.

For instance, since being a square (A) implies being a rectangle (B), we can say that being a square is sufficient for being a rectangle and being a rectangle is necessary for being a square. "Sufficient" is a useful word because it makes clear that A need not be the only thing which guarantees B – squares are not the only rectangles – but being a square suffices. "Necessary" is a useful word because it makes clear that B may not be enough to get A – being rectangular is only part of being square – but you can't do without it.

Since the usual goal in mathematics is to prove one thing from another, the value of knowing that A is sufficient for B is clear: Once you get A you are home free with B. But suppose instead you know that A is necessary for B. The implication is going the wrong way as far as deducing B is concerned, so what good is this knowledge of necessity? The answer is: The concept of necessity is natural and useful when you are after negative information. That is, suppose you know that A is necessary for B and you also know that A is *false*. Then you can conclude that B is false, too. In effect, you are using the contrapositive $\neg A \Longrightarrow \neg B$.

Definitions and Equivalence

If both $A \Longrightarrow B$ and $B \Longrightarrow A$ are true, we abbreviate by writing

$$A \iff B.$$

Such a statement is called a **biconditional** or **bi-implication**. We also say that statements A and B are (logically) **equivalent**. Two other ways to say $A \Longleftrightarrow B$ without symbols are

A is necessary and sufficient for B,

and

A if and only if B.

There is yet another accepted shorthand for "if and only if": iff. Should you need to pronounce iff, really hang on to the "ff" so that people hear the difference from "if".

Equivalence is a very useful concept. Two statements are logically equivalent if they are always both true or always both false. In other words, wherever we have said that two statements mean the same thing, we could instead have said they are equivalent. For instance, an implication and its contrapositive are equivalent. Symbolically,

$$(A \Longrightarrow B) \Longleftrightarrow (\neg B \Longrightarrow \neg A).$$

A **definition** is a declaration that henceforth a certain term will mean the same as the phrase that follows the term. For instance,

> Definition. A square is a four-sided figure with all sides equal and all angles 90°.

That was a **displayed** definition. You could also write an **in-line** definition: A square is defined to be a four-sided figure with all sides equal and all angles 90°.

Either way, a definition is always an equivalence and could be rephrased as such: A figure is a square iff it is four-sided with all sides equal and all angles 90°. Logically speaking, it would be incomplete to write "if" instead of "iff" here, but we regret to inform you that, by tradition, this is exactly what is usually done. This substitution of "if" when "iff" is meant is allowed *only* in definitions. You can always tell in this book when a sentence is a definition: either the word "define" is present, or else the term being defined is printed in boldface.

Once a term is defined, it becomes a **reserved word**. It can no longer be used in the mathematical parts of your document with any other or broader meanings which it has in ordinary English. It is easy to avoid such conflict with most defined math terms; they are words that rarely occur in ordinary English.

There are some exceptions. "Equivalent" is one. In ordinary English it often means "equal". However, in mathematics, "equal" and "equivalent" are *not* the same. For instance, the statements $A \Longrightarrow B$ and $\neg B \Longrightarrow \neg A$ are equivalent but not equal, since they are not identical. There are several other uses of equivalent in mathematics in addition to logically equivalent, but none of them mean equal.

In ordinary language, words are rarely defined but are understood through usage. In mathematics they are defined because added precision is important. A slight difference in what a word means can make the difference between whether a statement using that word is a theorem or not. Still, usage can help you understand defined terms. Whenever you read a definition, try to think up examples of things that meet the definition, and other things that don't but almost do. For instance, upon reading the definition of "square", first try to picture several different things that are squares – you'll see you can only make them differ in size. Then for contrast try to picture something that has four equal sides but not four right angles; then something with four sides and four right angles, but not all sides equal.

Quantifiers

Mathematicians are rarely interested in making statements about individual objects (2 has a square root); they prefer to make statements about large sets of objects (all nonnegative numbers have square roots). Words which indicate the extent of sets are **quantifiers**. Here are some true statements illustrating the use of the key quantifiers in mathematics. The quantifiers are in boldface.

Every integer is a real number.

All integers are real numbers.

Any integer is a real number.

Some real numbers are not integers.

There exists a real number which is not an integer.

No irrational number is an integer.

"Every", "all", and "any" are called **universal** quantifiers. "Some" and "there exists" are **existential** quantifiers.

Sometimes a quantifier is not stated explicitly but is understood, i.e., *implicit*. For instance, the statement

Primes are integers

or even

A prime is an integer

mean "All primes are integers." Implicit quantifiers are especially common for algebraic statements. If a text displays

$$x + y = y + x,$$

it is asserting that the order of addition is immaterial *for all* x and y. Implicit quantifiers are almost always universal quantifiers.

"Some" has a somewhat special meaning in mathematics. Whereas in ordinary discourse it usually means "at least a few but not too many", in mathematics it means "at least one, possibly many, or possibly all". "There exists" has exactly this same meaning. Thus the following peculiar mathematical statements are technically correct, as explained just below:

Some even integers are primes.

Some positive numbers have square roots.

There exists a positive number with a square root.

The first statement is correct because there is exactly one even prime: 2. The other two are correct because every positive number has a square root.

Why "some" is used this way will be explained in the subsection on negation. For now let's admit that mathematicians would rarely write the three statements displayed above. It takes no more space to tell the whole story, so we say,

There is exactly one even prime: 2.

and

All positive numbers have square roots,

or even better,

A real number has a real square root $\Longleftrightarrow$ it is nonnegative.

However, if all we know is that at least one "smidgit" exists, and we have no idea how many there are, then we do not hesitate to write "There exists a smidgit" (even though there might be many) or "There are some smidgits" (even though there might be only one).

Another way to state that there is just one of something is to use "the", with emphasis. Suppose $x = 3$ is the only solution to some problem. Then say "$x = 3$ is *the* solution" (pronounced *thee*). If there is more than one solution, say "$x = 3$ is *a* solution" (pronounced like capital A) or "*an* answer". Thus, 2 is the even prime and 5 in an odd prime.

Differences among universal quantifiers. We have not drawn any distinctions between "every", "any", and "all". However, in English usage there are sometimes differences, especially between "every" and "any". For instance, you will surely agree that the following sentences mean the same:

Everybody knows that fact.

Anybody knows that fact.

But just as surely the following sentences don't mean the same thing:

She'll be happy if she wins every election.

She'll be happy if she wins any election.

In general, "any" is a very tricky word, meaning "all" in most instances but "one" in some [12].

Why then don't mathematicians simply avoid "any" and always use "every" or "all"? Perhaps the reason is that "any" highlights a key aspect of how to prove statements about an infinite number of objects in a finite number of words. Suppose we wish to prove that "Any square is a rectangle." The word "any" suggests, correctly, that it suffices to pick *any one* square and show that *it* is a rectangle – so long as we don't use any special knowledge about that square but only its general squareness. See Example 1 later. The words "every" and "all" don't suggest this approach quite as well.

Proving for all by proving for any one is sometimes called arguing from the **generic particular**. In such a proof, it is important to check that the example really is generic – it cannot have special features, or if it does, they cannot be used in the proof.† As you read through our proofs in this text, check to see that we have

† Having completely general features is what "generic" meant long before it referred to discount drugs.

paid attention to this requirement. The discussion of buildup errors in Section 2.8 is particularly relevant to this concern.

Another frequently used word for generic is *arbitrary*, as in "consider an arbitrary square".

In Section 7.6, we'll discuss the subject of quantifiers in a more formal mathematical setting and show that many aspects of using them correctly can be handled mechanically.

Negation

The **negation** of statement A is a statement which is true precisely when A is false. For simple statements, it is very easy to create the negation: just put in "not". For instance,

The negation of "x is 2" is "x is not 2."

Thus we have been using negations all along whenever we wrote (not A) or $\neg A$.

You should be warned though that negating can get much trickier if the original statement contains quantifiers or certain other words. For instance, the negation of

Some of John's answers are correct

is not

Some of John's answers are not correct,

because it is not the case that the latter statement is true if and only if the former is false. They are both true if John gets half the questions right and half wrong. The correct negation of the first sentence is

None of John's answers is correct.

We said earlier we would explain why "some" is used in mathematics to mean "one up to all". The reason is so that it will mean precisely the negation of "none". If "some" meant "a few", it would only cover part of the situations included in the negation of "none". Similarly, if "there exists" was limited to meaning "exactly one exists", then it, too, would not negate "none".

Theorems and Proofs

Three names are commonly used for proven mathematical statements. A **theorem** is an important proven claim. A **lemma** is a proven claim which, while not so important in itself, makes the proof of a later theorem much easier. A **corollary** is a result which is quite easy to prove given that some theorem has already been proved. In other words, lemmas help to prove theorems, and theorems imply corollaries.

Strictly speaking, lemmas are unnecessary. The statement of a lemma, and its proof, could be incorporated into the proof of the theorem it is intended to help. However, including the lemma may make the proof of the theorem so long that the

key idea is obscured. Also, if a lemma is helpful for several different theorems, it clearly is efficient to state it separately.

A **conjecture** is a statement for which there is evidence or belief, but which has not been proved. Interestingly, in other sciences such a statement is instead called an hypothesis. (It is currently an hypothesis that the primary cause of global warming is human-produced hydrocarbons.) However, with rare exceptions mathematicians reserve the word "hypothesis" for the if-clause of implications, especially implications that are theorems. (The hypothesis of the Pythagorean Theorem is that a and b are the lengths of the legs of a right triangle and c is the length of the hypotenuse.)

But what is a proof? An air-tight logical demonstration that a claim is true. Of course, saying this doesn't help much, because there many varieties of logical arguments. For now, we want you to recognize three key formats of proofs: direct, indirect, and proof by contradiction.

Suppose, as usual, that we are trying to prove $A \Longrightarrow B$. A **direct proof** starts with the hypothesis A and ends (after carefully using implications!) with the conclusion B. An **indirect proof** starts instead with the assumption that $\neg B$ is true (that is, not-B). How can such a start prove B? One way is by ending up with $\neg A$. That is, you prove the theorem in the contrapositive form. A **proof by contradiction** is a special sort of indirect proof in which you end up with an obvious impossibility (say, $1 = 0$), so the assumption $\neg B$ must be false.

EXAMPLE 1 Give two proofs, one direct and one indirect, for the theorem that every square is both a rhombus and a rectangle.

Solution First we must state this theorem as an implication, so that we know what we may assume and where we have to go. We may take the implication to be $A \Longrightarrow B$, where A is the statement "A quadrilateral is a square" and B is "A quadrilateral is a rhombus and a rectangle." Recall that, by definition, a rhombus is a 4-sided figure with all sides equal in length, and a rectangle is a 4-sided figure with four 90° angles.

Direct proof:

> Let Q be a square. By definition, Q has four equal sides and four 90° angles. Because it has four equal sides, it is a rhombus. Because it has four 90° angles, it is a rectangle. Thus Q is a rhombus and a rectangle.

Indirect Proof:

> Suppose Q is not both a rhombus and a rectangle. If it is not a rhombus, then some pair of sides are not equal. If it is not a rectangle, then some angle is not a right angle. Either way Q is not a square.

In the indirect proof, we have indeed proved the contrapositive, $\neg B \Longrightarrow \neg A$. ∎

In both proofs, Q is a generic particular. This is particularly clear in the direct proof, because of the phrase "a square".

Also, both proofs in Example 1 were *from the definitions*, meaning you merely had to substitute the definitions of the key terms, be orderly, and you were done. Not all proofs work this way, but the first few after a new concept is introduced usually do.

EXAMPLE 2 Use proof by contradiction to prove that there are no even primes greater than 2.

Solution Here is such a proof.

> Suppose not (meaning, suppose the conclusion is false and thus there exists an even prime $n > 2$). By definition of even, $n = 2m$ for some integer $m > 1$ since $n > 2$. Thus n is not a prime since 2 is a factor. Contradiction! ∎

Indirect proofs are particularly well suited to **nonexistence** theorems, as in Example 2. If you are trying to show that an object with certain properties does not exist, assume that it does and strive for a contradiction. By assuming it does, at least you have an object to play around with.

Any indirect proof by contrapositive can be turned into a proof by contradiction by adding one more step. To prove $A \Longrightarrow B$, you supposed $\neg B$ and reached $\neg A$. Then add that, by hypothesis, A is also true. Therefore A and $\neg A$ are both true – a contradiction. However, you need not add that additional step. A proof by contrapositive stands on its own as a type of indirect proof.

Proofs of equivalences. In Example 1 you probably noticed that more is true: a quadrilateral is a square *if and only if* it is a rhombus and a rectangle. The simplest way to prove $A \Longleftrightarrow B$ is to have two parts, a proof of $A \Longrightarrow B$ and a proof of $B \Longrightarrow A$. Each part can be direct or indirect [15]. At the start of each part, be sure to tell the reader which part you are doing.

Other types of proofs. There are many ways to classify proofs. An important distinction in this book is between existential proofs and constructive proofs. Constructive proofs have an important subcase, proofs by algorithm. These three categories are discussed by examples beginning on p. 252. Another important type of proof is by mathematical induction, the subject of Chapter 2. All these types can involve either direct or indirect arguments; see [13, Section 2.2].

Justification below the proof level. While proof is the standard of justification in mathematics, lesser types of evidence have their place. What do we mean if we ask you to "verify" or "show" something instead of prove it?

It is a fact (shown in Chapter 4) that each n-set has exactly 2^n subsets. Suppose we ask you to verify this in the case of $n = 2$ or show it for the set $S = \{a, b\}$. For the latter request you would just write down all subsets of S and count that there are 4. For the former request you might say that it suffices to consider this same set S and again list its subsets. In short, for an example "verify" and "show" mean check it out in that case by any method, including brute force.

If instead we simply say "show that an n-set has exactly 2^n subsets", we are asking you to try to come up with a general argument, but we won't be too picky.

Since we haven't discussed yet the key counting principles (to come in Chapter 4), we can't expect you to write up your reasoning very formally. Moreover, if you don't see any general argument, it's OK to show what evidence you can. You could write out all the subsets for $n = 1, 2, 3$ and observe that the number is indeed doubling each time.

Sometimes we leave steps out of a proof in the text. If we then say "verify the details", that means fill them in.

Trivial, Obvious, Vacuously True

Many people think "trivial" and "obvious" mean the same thing. However, writers of mathematics use them differently. An argument is **trivial** if every step is a completely routine calculation (but there may be many such steps). A claim is **obvious** if its truth is intuitively clear (even though providing a proof may not be easy). For instance, it's trivial that

$$(x+2)(3x+4)(5x^2+6x+7) = 15x^4 + 68x^3 + 121x^2 + 118x + 56,$$

but it's hardly obvious. On the other hand, it's obvious that space is three-dimensional but the proof is hardly trivial.

Students sometimes think that writers or lecturers who use these words are trying to put them down, and perhaps sometimes they are. But, used appropriately, these words are useful in mathematical discourse and need not be threatening.

Sometimes a claim is made about something that doesn't exist. Then mathematicians consider that claim to be **vacuously true**. How could such claims be of any interest and why declare them to be true?

Here is the archetype example. Let S be any set. Is the empty set $\emptyset$ a subset? Well, by definition of subset, we have to check whether

Every element of $\emptyset$ is an elementof S.

Since $\emptyset$ has no elements and thus provides no examples to contradict the claim, mathematicians declare the claim to be vacuously true.[†]

Now, one doesn't naturally think about the empty set when looking at subsets, so why it is useful to insist that it is a subset? Well, if $A \subset S$ and $B \subset S$, should it be true that $A \cap B \subset S$? Of course; this is obvious from a Venn diagram [20]. But what if $A \cap B = \emptyset$? If the empty set wasn't a subset of every set, then this little theorem about the intersection of subsets would have an exception. Or rather, it would have to be stated more carefully: *nonempty* intersections of subsets are subsets. It turns out that by declaring vacuous statements to be true, all sorts of annoying little exceptions are avoided.

Here is another example. Suppose we have an algorithm for taking a list of numbers and putting them in increasing order. In order means: for any two distinct elements x, y of the list, if $x < y$, then x precedes y. Notice that this definition

[†] Technically speaking, we are considering the implication $x \in \emptyset \Longrightarrow x \in S$, in which the hypothesis is false for every x. An implication $A \Longrightarrow B$ is considered to be true whenever A is false, as well as when A and B are true. All this is discussed further in Chapter 7.

is vacuous if the list has only 0 or 1 elements – there are no distinct pairs x, y. Suppose we want to state a theorem that our algorithm works correctly. If vacuous statements were considered to be false or meaningless, our theorem would have to say that the algorithm works *when given a list of at least two numbers*. However, with vacuous statements declared to be vacuously true, we don't have to include the italicized caveat.

Finally, sometimes a claim is true not because it is about nothing, but because nothing has to be done to prove it. Such claims are also, sometimes, called vacuously true. Here is a common situation. Suppose an algorithm takes some positive number and does some operation repeatedly, say squaring, and we want to argue that the result is still positive. A standard approach is to restate the claim as "After squaring k times, the result is still positive." Next we would proceed to prove the claim for all k by the method of induction (Chapter 2). We often choose to start with the case $k = 0$, because it is easy. The original number is "still" positive after squaring 0 times because nothing has been done to it yet. The number exists, but one sometimes says that it is vacuously true that the number is still positive.

And and Or

In mathematical expressions "and" may be implicit. For instance, both

$$A = \{(x, y) \mid x > 0,\ y > 0\} \qquad \text{and} \qquad A = \{(x, y) \mid x, y > 0\}$$

mean the set of all points (x, y) with x *and* y greater than 0, i.e., the first quadrant. If you want x *or* y greater than 0, you have to say so explicitly:

$$C = \{(x, y) \mid x > 0 \text{ or } y > 0\}. \tag{1}$$

Be careful about "or". In English it is often used to mean that exactly one of two possibilities is true:

Either the Democrats or the Republicans will win the election. (2)

On other occasions "or" allows both possibilities to hold:

You can get there via the superhighway or by local roads.

The first usage is called the **exclusive or**, the second the **inclusive or**. In mathematics, "or" is always inclusive unless there is an explicit statement to the contrary. Thus C in Eq. (1) is all points except those in the third quadrant (more precisely, except those with $x, y \le 0$). If for some strange reason statement (2) was the object of mathematical analysis, it would have to be prefaced by "exactly one of the following holds" or followed by "but not both".

Problems: Section 0.6

1. ⟨1⟩ Restate each of these in if-then form:
   a) If it's Tuesday, this must be Belgium.
   b) When it's hot, flowers wilt.
   c) To snow it must be cold.

2. ⟨1⟩ Explain the meaning of the words
   a) implication b) hypothesis
   c) equivalence d) negation.

3. ⟨2⟩ Find an implication which is
   a) true but its converse is false;
   b) false but its converse is true;
   c) true and its converse is true;
   d) false and its converse is false.

4. ⟨2⟩ Which of the following statements are true? Why?
   a) To show that $ABCD$ is a square, it is sufficient to show that it is a rhombus.
   b) To show that $ABCD$ is a rhombus, it is sufficient to show that it is a square.
   c) To be a square it is necessary to be a rhombus.
   d) A figure is a square only if it is a rhombus.
   e) A figure is a square if and only if it is a rhombus.
   f) For a figure to be a square it is necessary and sufficient that it be an equilateral parallelogram.

5. ⟨2⟩ State the converse for the following.
   a) If $AB \parallel CD$, then $ABCD$ is a parallelogram.
   b) If a computer program is correct, it will terminate.

6. ⟨2⟩ State the contrapositive for the sentences in [5].

7. ⟨3⟩ The *inverse* of $A \Longrightarrow B$ is the statement $\neg A \Longrightarrow \neg B$.
   a) Give an example to show that a statement can be true while its inverse is false.
   b) Give an example to show that a statement can be false while its inverse is true.
   c) What statement related to $A \Longrightarrow B$ is its inverse logically equivalent to? Explain why.

8. ⟨2⟩ Which of the following statements are logically equivalent? Why?
   (i) $x = 3$. (ii) $x^2 = 9$. (iii) $x^3 = 27$.
   (iv) x is the smallest odd prime.
   (v) $|x| = 3$.

9. ⟨2⟩ Find a sentence in this section where we used the convention that in a definition "if" may be used where "iff" is meant.

10. ⟨2⟩ Which of the following are true? Interpret the statements using the conventions of mathematical writing.
    a) Some rational numbers are integers.
    b) Some equilateral triangles are isosceles.
    c) Some nonnegative numbers are nonpositive.
    d) Some negative numbers have real square roots.
    e) There exists an odd prime.
    f) There exists a number which equals its square.
    g) Either $\pi < 2$ or $\pi > 3$.
    h) Not all primes are odd.

11. ⟨2⟩ Which pairs from the following statements are negations of each other?
    (i) All mammals are animals.
    (ii) No mammals are animals.
    (iii) Some mammals are animals.
    (iv) Some mammals are nonanimals.
    (v) All mammals are nonanimals.
    (vi) No mammals are nonanimals.

12. ⟨2⟩ In some of the following sentences, "every" can be replaced by "any" without changing the meaning. In others, the meaning changes. In still others the replacement can't be made at all — the resulting sentence isn't good English. For each sentence, decide which case applies. (When it comes to what is good English, there may be disagreements!)
    a) He knows everything.
    b) Everybody who knows everything is to be admired.
    c) Everybody agreed with each other.
    d) Not everybody was there.
    e) Everybody talked at the same time.
    f) Every square is a parallelogram.

13. ⟨2⟩ There is always a way to negate correctly a sentence using *not*. What is it? (*Hint*: Use "not" as part of a phrase at the beginning of the sentence.)

14. ⟨2⟩ Is the following a valid proof that "an integer is divisible by 9 if the sum of its digits is divisible by nine"?

> Proof: Consider such a number n in the form $10A + B$. Since $10A + B = 9A + (A + B)$ and since $9\,|\,(A + B)$ by hypothesis, then $9\,|\,n$.

The next several problems ask you to prove things. Many of the things are obvious, and we wouldn't normally ask for written proofs. The point is to practice proof styles on toy examples.

15. ⟨2⟩ Example 1 proved half of the theorem "A quadrilateral is a square $\Longleftrightarrow$ it is a rhombus and a rectangle. Prove the other half
    a) by a direct argument,
    b) by an indirect argument.

16. ⟨2⟩ Prove that there is no biggest integer by proving "For every integer n there is a bigger integer." Start by picking *any* integer n and naming a bigger integer using the letter n. (By using n, rather than an example, your argument is generic.)

17. ⟨3⟩ Let O be the set of odd numbers, and let S be the set of all integers that may be expressed as the sum of an odd number and an even number.
    a) Show that $S \subset O$. *Hint*: By definition of subset, we must show that every element of S is an element of O. So pick a generic particular number in S and show that it is odd.
    b) Show that $S = O$. One standard method for doing this is to show both $S \subset O$ and $O \subset S$. Thus, you need a second generic particular argument beyond that in a).

18. ⟨2⟩ Prove that there does not exist a number that solves $x = x + 1$. Remember, contradiction is usually the best way to prove nonexistence.

19. ⟨3⟩ Prove in several styles that $3x+4 = 10$ if and only if $x = 2$. Use the labels "If" and "Only if"; use arrows; do direct proofs; do indirect proofs.

20. ⟨2⟩ Show with a Venn diagram that if $S, T \subset U$ then $S \cap T \subset U$.

21. ⟨2⟩ Give a written proof that

$$[S, T \subset U] \implies [S \cap T \subset U].$$

Start by picking a generic element in $S \cap T$; show it is in U.

22. ⟨2⟩ Can the theorem in [21] be strengthened by weakening the hypothesis? Suppose only S is known to be a subset of U; T can be any set. Must $S \cap T$ still be subset of U? Give a proof or **counterexample** – an example that shows the claim to be wrong.

23. ⟨2⟩ Which of the following are vacuously true; which not? Explain. When a statement is not vacuous, is it trivial? Obvious? Explain briefly.
    a) For all integers k such that $1 \le k \le 0$,
    $$2k^3 + 2k^2 - 3k = 1.$$
    b) For all integers k such that $1 < k \le 2$,
    $$k^9 - 3k^7 + 4k^5 - 30k^3 = 16.$$
    c) Every prime which is a perfect square is greater than 100.
    d) The case $n = 2$ of "The sum of the degrees of the interior angles of every planar n-gon is $180(n-2)$."
    e) The same proposition as part d) when $n = 3$.
    f) The case $n = 1$ of: If all $a_i > 0$, then
    $$\left(\sum_{i=1}^{n} a_i\right)^2 \ge \sum_{i=1}^{n} a_i^2.$$

Supplementary Problems: Chapter 0

1. ⟨2⟩ The **greatest common divisor**, $\gcd(m, n)$, of two integers is the largest positive integer which can be divided into both without leaving a remainder. In Chapter 1 we'll derive a method for computing the greatest common divisor of two integers. For this problem, however, just use any (brute force) method you wish. Find the gcd of the following pairs of integers.

   a) 315 and 91 b) 440 and 924

2. ⟨2⟩ The **least common multiple**, $\mathrm{lcm}(m, n)$, of two nonzero integers is the smallest positive integer into which both m and n can be divided without leaving a remainder. Find the lcm of the following pairs of integers.

   a) 6 and 70 b) 33 and 195

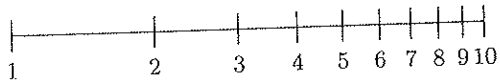
3. ⟨3⟩
   a) Find the greatest common divisor of each pair of integers in [2].
   b) Using the results of part a) and those of [2], conjecture what the product of the gcd and lcm of two integers always is.
   c) Try to prove your part b) conjecture.

4. ⟨2⟩ The function $\mathrm{loglog}\, x$ is defined by $\mathrm{loglog}\, x = \log(\log x)$.
   a) What is the domain of loglog? The codomain?
   b) Show that $\mathrm{loglog}(c^k) = \log k + \mathrm{loglog}\, c$.

5. ⟨3⟩ For some time after logarithms were invented, scientists did computations with logarithms by looking them up from books of tables. Long ago the slide rule eliminated the need to use logarithms for computation except where accuracy to more than 2 or 3 decimal places was needed. Only in the last quarter of the 20th century did hand calculators eliminate the slide rule.

   This problem is for people who can find an old slide rule in the attic and want to figure out how it worked.

   a) Numbers were multiplied on slide rules by moving two "log scales" relative to each other. On a log scale, each number x is marked at the distance $k \log_{10} x$ from the left of the scale, where k is the length of the entire scale (see the scale in the figure). Explain how Property 1 allows us to multiply numbers by using such scales.



   Since $\log_{10} 3 = .477$, the point labeled 3 is .477 of the way from 1 to 10.

   b) Explain how you could find c^k by moving a "loglog scale" relative to a log scale. *Hint*: Use the result from [4].

6. ⟨3⟩ A value c is called a **fixed point** of the function f if $f(c) = c$.
   a) Find all the fixed points of the function $f(x) = 3x + 8$.
   b) Find all the fixed points of the function $f(x) = |x|$.
   c) Let $U = \{1, 2, 3, 4\}$ and let f be defined on subsets of U by $f(A) = U - A$. Find all the fixed points of f. (*Note*: In this case "points" means "set".)
   d) Let f have as its domain the set of strings of lowercase roman letters, and let f map each string into its reverse. For instance, $f(xyz) = zyx$ and $f(\mathrm{love}) = \mathrm{evol}$. Name a fixed "point" of length 3. Name a fixed point of length 4.

7. ⟨3⟩ Let
$$H_n = \sum_{k=1}^{n} \frac{1}{k}.$$
   Let
$$O_n = \sum_{\substack{k=1 \\ k \text{ odd}}}^{n} \frac{1}{k}.$$
   For instance,
$$O_{10} = \frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9}.$$
   Show that $O_n = \mathrm{Ord}(H_n)$.

8. ⟨4⟩ Find a general formula for
$$\sum_{S \subset [n]} |S|,$$
   with $[n] = \{1, 2, \ldots, n\}$. *Hint*: You can write th as the double sum $\sum_{S\subset[n]} \sum_{i \in S} 1$; then try chan ing the order of summation.

9. ⟨3⟩ Suppose that you have two algorithms for the same task. Let the execution time (in seconds) for the first algorithm be $n^2/10 + n/5 + 1/3$ and let that for the second be $50n \log_2 n$.

a) Calculate the execution time for each algorithm for $n = 2^k$, $k = 0, 1, 2, \ldots, 15$.

b) Suppose that you knew that you had to do the task for which these two algorithms were written for 10 different but unknown (because they might depend on some other, not yet performed calculation) values of n which, with equal probability, might be any value from 1 to 50,000. Which algorithm would you choose?

10. ⟨3⟩ Let $f(x) = x^2 - 3x + 2$.

a) Show that $f(-x) = x^2 + 3x + 2$.

b) Show that if c is a root of
$$x^2 - 3x + 2 = 0,$$
then $-c$ is a root of
$$x^2 + 3x + 2 = 0.$$

c) Generalize part a). If
$$f(x) = \sum_{k=0}^{n} a_k x^k,$$
what is $f(-x)$? (If you can't figure out how to write this with summation notation, write it with dots.)

d) Generalize part b). If $f(c) = 0$, what equation does $-c$ solve?

11. ⟨3⟩ Let
$$f(x) = \sum_{k=0}^{n} a_k x^k,$$
where all the a_k are integers.

a) Show: If 0 is a root of $f(x) = 0$, then $a_0 = 0$.

b) Show: If $f(c) = 0$, where c is an integer, then $c|a_0$. (*Hint*: Subtract a_0 from both sides of the identity $f(c) = 0$.)

c) Show: If $f(b/d) = 0$, where b/d is in lowest terms, then $d|a_n$ and $b|a_0$.

d) What are the only *possible* rational roots of $2x^2 - x + 6 = 0$? That is, answer the question *without* solving the equation.

12. ⟨3⟩

a) Verify that the zeros of $2x^2 - x - 6$ are the reciprocals of the zeros of $-6x^2 - x + 2$.

b) Show: If $x \neq 0$ is a root of
$$\sum_{k=0}^{n} a_k x^k = 0,$$
then $1/x$ is a root of
$$\sum_{k=0}^{n} a_{n-k} x^k = 0.$$

13. ⟨3⟩ Descartes's **rule of signs** says that the number of positive zeros of the polynomial $P_n(x)$, as given by Eq. (6) of Section 0.2, is the number of variations in sign (from plus to minus and vice versa) in the sequence
$$a_0, a_1, \ldots, a_n$$
(with zeros ignored) or less than this by an even number.

a) State an analogous rule for the number of negative zeros.

b) Deduce a rule for the number of real zeros.

14. ⟨2⟩ Apply Descartes's rule and your results in [13] to find an upper bound on the number of positive, negative, and real zeros of

a) $P_5(x) = x^5 + 3x^4 - x^3 - 7x^2 - 16x - 12$;

b) $P_{10}(x) = x^{10} + x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$;

c) $P_6(x) = x^6 - 3x^4 + 2x^3 - x + 1$.